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THE OBLIQUE REFLECTION OF A RAYLEIGH WAVE FROM A CRACK TIP

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Abstract—The reflection of a free-surface Rayleigh wave from the edge of a half-plane crack in an unbounded isotropic elastic solid is considered. The time–harmonic surface wave, propagating on one face of the crack, is obliquely incident on the edge so that the problem is three-dimensional. Application of a displacement representation theorem reduces solution of the problem to solution of an integral equation, which is solved by application of Laplace transform methods and the Wiener–Hopf technique. The transformed displacement has a simple pole corresponding to the reflected surface waves, and the residues are determined. The amplitudes and phases of the reflected surface waves on both faces of the crack have been calculated numerically, and are plotted as functions of the angle of incidence. Energies of the reflected waves are also shown in graphs.

INTRODUCTION

IT HAS been known for some time that elastic surface waves may be guided over large distances by forming waveguides on surfaces of elastic materials in certain ways. In a recent paper [1], a method was introduced by which approximate dispersion relations for a class of surface waveguides could be obtained. In order to apply the method, the results of an auxiliary reflection problem must be available. Each edge of a surface waveguide may be viewed as a discontinuity in surface impedance. The auxiliary problem which must be solved is then the reflection of a straight-crested surface wave from one edge of the waveguide, that is, from a discontinuity in surface impedance, assuming the other edge of the guide to be absent. The solution of this problem consists of the determination of the amplitude and phase of the reflected surface wave as a function of angle of incidence of the incident wave.

All surface waveguides which have been proposed are formed by modifying the plane surface of an elastic solid [2]. For example, a thin film of material having certain desirable characteristics may be deposited either inside or outside the region of the guide [3]. It appears that a slit (an open crack of finite width and indefinite length) running through the interior of an elastic solid can also act as a surface waveguide. The surface waves, of course, propagate on the faces of the slit along its length. In order to determine the dispersion relation of this guiding configuration, it is necessary to first study an auxiliary reflection problem, which is the purpose here. The problem considered is the oblique reflection of a surface wave, propagating on one face of a half-plane crack, from the edge of the crack. The study of the waveguide itself will be taken up in a subsequent paper. The simpler problem of a normally incident surface wave, which is a special case of the problem considered here, was studied in [4].

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SOLUTION FOR REFLECTED SURFACE WAVES

The problem of an unbounded homogeneous elastic solid containing a semi-infinite crack is considered. Cartesian coordinates $x_i = (x, y, z)$ are prescribed in the body in such a way that the crack occupies the half-plane $z = 0, x < 0$; see Fig. 1. The crack is assumed to remain open at all times, that is, the faces of the crack are taken to be free of traction. The index notation of Cartesian tensor analysis is used in the following analysis, and the range of the indices is usually 1, 2, 3.

In the absence of body forces, the equation governing the components of the displacement vector $U_i(x, t)$ is

$$C_{ijkl}U_{k,jl} - \rho \ddot{U}_i = 0, \qquad i = 1, 2, 3 \tag{1}$$

where ρ is mass density. For the isotropic material being considered, C_{ijkl} can be expressed in terms of the Lamé constants Λ and μ as

$$C_{ijkl} = \Lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2}$$

The boundary conditions which must be satisfied for the faces of the crack to be traction-free are

$$\Sigma_{i3}(x, y, 0^{\pm}, t) = 0, \qquad x < 0, \qquad i = 1, 2, 3 \tag{3}$$

where the notation $0^{\pm}$ means that the indicated stress components must vanish as z approaches zero through positive or negative values. The stress matrix is related to the displacements by

$$\Sigma_{ij} = C_{ijkl}U_{k,l}. \tag{4}$$

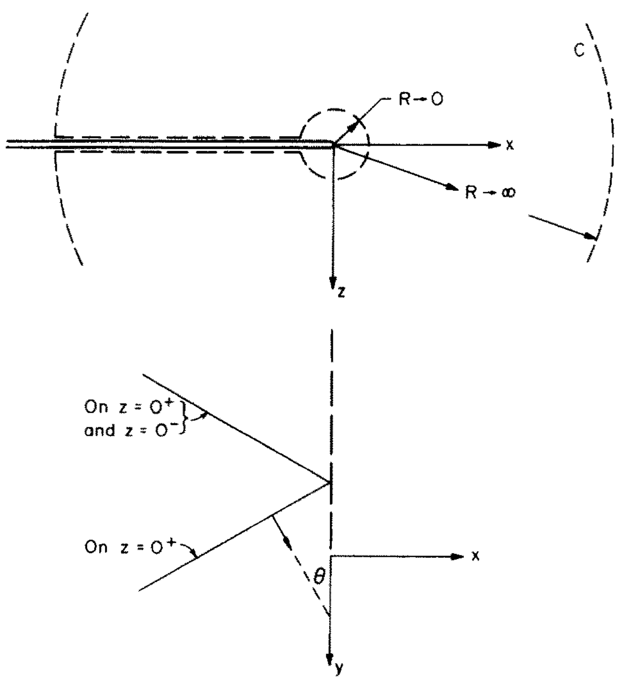

FIG. 1. The physical system viewed along the negative y-axis and the positive z-axis.

The excitation is in the form of a surface wave on the face of the crack $z = 0^+$. A steady-state situation is assumed to exist with the surface wave, harmonic in time with circular frequency ω, obliquely incident on the edge of the crack at $x = z = 0$. Making use of the notation for the discontinuity of a function across $z = 0$,

$$\Delta U_i(x, y, t) \equiv U_i(x, y, 0^+, t) - U_i(x, y, 0^-, t), \tag{5}$$

the discontinuity in the incident disturbance is written as

$$(\Delta U_i)^{\text{inc}} = A_i \, e^{i(\omega t - \alpha x - \beta y)}, \qquad x < 0 \tag{6}$$

where A_i is an amplitude, and α and β are components of the surface wavenumber vector. If γ is the wavenumber of Rayleigh waves, then $\alpha = \gamma \cos\theta$ and $\beta = \gamma \sin\theta$. Any two components of A_i may be expressed in terms of the third, for example,

$$\begin{aligned} \beta A_1 &= \alpha A_2, \qquad \alpha A_1 + \beta A_2 = A_3 \Gamma \\ \Gamma &= -2i\gamma^2(\gamma^2 - \kappa_b^2)^{\frac{1}{2}}/(2\gamma^2 - \kappa_b^2) \end{aligned} \tag{7}$$

where κ_b is the wavenumber for shear waves. Therefore, once the frequency and direction of propagation of a Rayleigh wave are specified, the wave can be completely characterized by a single amplitude. (Since amplitudes here are taken to be complex, this means two real quantities must be specified, for example, a real amplitude and a real phase angle.)

It is assumed that the scattered field has the same harmonic time-dependence as the incident wave. Furthermore, the fact that the physical system is invariant with respect to translation in the y-direction makes it possible to write the dependence of displacement and stress on y in the form

$$U_i(x, y, z, t) = u_i(x, z) \, e^{i(\omega t - \beta y)}, \tag{8}$$

$$\Sigma_{ij}(x, y, z, t) = \sigma_{ij}(x, z) \, e^{i(\omega t - \beta y)}, \tag{9}$$

$$\Delta U_i(x, y, t) = \Delta u_i(x) \, e^{i(\omega t - \beta y)}. \tag{10}$$

The problem is thus reduced to the determination of the amplitudes u_i and σ_{ij} in the x, z-plane.

A very useful result in solving for these amplitudes is a displacement representation theorem due to de Hoop [5]. In Section 4 of [5] the problem of an elastic solid bounded by a closed surface is considered. Time-dependent boundary data are specified on the surface in such a way that the elastodynamic problem is well-posed. It is then shown that the displacement at any interior point and at any time can be represented as a sum of two surface integrals over the bounding surface, plus a body force term which is neglected here. A similar representation was derived by Krupradze in chapter 1 of [6] for steady-wave problems.

A representation theorem of use in solving the problem at hand may be derived from the result in [5]. Since this representation is valid for any time-dependence, the simple harmonic time-dependence may be made explicit and some integrals over time may be evaluated. Suppose that the elastic solid is a long cylinder whose lateral surface is generated by a line parallel to the y-axis moving along a curve C in the x, z-plane. The surface of the body is then made up of the cylindrical surface plus the region of the planes $y = \pm L$ interior to C, where L is large. In view of (8)–(10), the dependence of all fields on y may then be made explicit in the representation formula, and the integration with respect to y may be performed. The result of this tedious but straightforward calculation is, for $L \to \infty$,

$$u_i(\xi) = \int_C C_{jklm}\Gamma_{ij}(\xi, s)u_{l,m}(s)n_k(s)\,\mathrm{d}s + \frac{\partial}{\partial x_m}\int_C C_{jklm}\Gamma_{il}(\xi, s)u_j(s)n_k(s)\,\mathrm{d}s \tag{11}$$

where $\xi = (x, z)$ is any point interior to C, $s = (x_s, z_s)$ is a coordinate along C, n_k is the outward normal to C, and

$$\Gamma_{ij}(\xi, s) = \frac{1}{2\pi\omega^2}\left\{-\frac{\partial^2}{\partial x_i \partial x_j} + i\beta\delta_{i2}\frac{\partial}{\partial x_j} + i\beta\delta_{j2}\frac{\partial}{\partial x_i} + \beta^2\delta_{i2}\delta_{j2}\right\} \cdot \{K_0(\lambda_a r) - K_0(\lambda_b r)\} + \kappa_b^2\delta_{ij}K_0(\lambda_b r), \tag{12}$$

$$r = [(x_s - x)^2 + (z_s - z)^2]^{\frac{1}{2}}. \tag{13}$$

In (12), K_0 is the modified Bessel function of the second kind, κ_a is the wavenumber of dilatational waves, and

$$\lambda_a = (\beta^2 - \kappa_a^2)^{\frac{1}{2}}, \qquad \lambda_b = (\beta^2 - \kappa_b^2)^{\frac{1}{2}}. \tag{14}$$

For the problem being considered here, the curve C is shown in Fig. 1. In view of the boundary conditions (3), the representation (11) reduces to

$$u_i(x, z) = -\frac{\partial}{\partial x_m}\int_{-\infty}^{0} C_{j3lm}\Gamma_{ij}(x, z, x_s, 0)\Delta u_j(x_s)\,\mathrm{d}x_s. \tag{15}$$

Relation (15) has exactly the same form as the representation formula derived in [4]. The array Γ_{ij} is quite different in this case, however, because (15) is a special case of a three-dimensional representation while equation (12) of [4] is strictly a two-dimensional result.

The general scheme for obtaining the reflected surface waves on the faces of the crack, due to the incident surface wave, is the following. A bilateral Laplace transform is defined over the spatial variable x. This transform is then applied to the representation formula (15), to the stress–displacement relation (4), and to the boundary conditions (3). Then, working exclusively with transformed quantities, the representation formula is substituted into the stress–displacement relation, and the boundary conditions are imposed. The result of these steps is a system of Wiener–Hopf type equations. In general, such systems of functional equations cannot be solved [7]. For the particular case being studied here, however, the system is degenerate and the equations can be combined in such a way as to yield three uncoupled Wiener–Hopf equations of the standard type.

The bilateral Laplace transform of a function is denoted by a bar over the function and is defined by

$$\bar{\phi}(\lambda, z) = \int_{-\infty}^{\infty} \mathrm{e}^{-\lambda x}\phi(x, z)\,\mathrm{d}x. \tag{16}$$

The same reasoning employed in [1, 4] suggests that the inversion path of integration for (16) is the imaginary axis, approached from the right in the lower half-plane and from the left in the upper half-plane, as shown in Fig. 2 for the case $\kappa_b > \beta > \kappa_a$. Furthermore, transforms of functions which represent outgoing waves and which vanish on the half-line $x > 0\,(x < 0)$ are taken to be analytic on, and to the left (right) of, the inversion path.

The transform is first applied to (15), making use of the familiar result for transforming a convolution integral. The result is

$$\bar{u}_i(\lambda, z) = G_{ij}(\lambda, z)\overline{\Delta u_i}(\lambda), \tag{17}$$

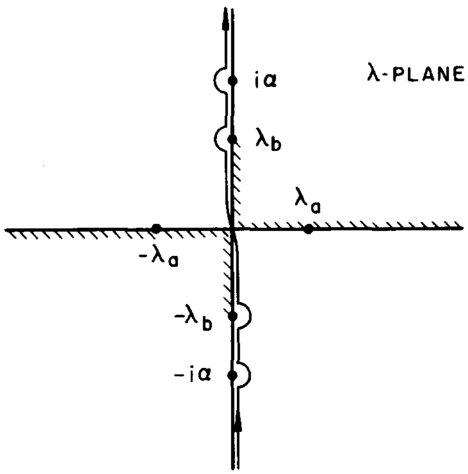

FIG. 2. The complex λ-plane for the case when λ_a is real and λ_b is imaginary, including the transform inversion path.

where

$$G_{ij}(\lambda, z) = -\int_{-\infty}^{\infty} e^{-\lambda x} C_{j3lm}\Gamma_{il,m}(x, z, 0, 0)\,dx \tag{18}$$

$$\overline{\Delta u_i}(\lambda) = \int_{-\infty}^{0} e^{-\lambda x}\Delta u_i(x)\,dx. \tag{19}$$

Application of the transform to the stress–displacement relation yields

$$\bar{\sigma}_{13}/\mu = \bar{u}_{1,3} + \lambda\bar{u}_3 \tag{20}$$

$$\bar{\sigma}_{23}/\mu = \bar{u}_{2,3} - i\beta\bar{u}_3 \tag{21}$$

$$\bar{\sigma}_{33}/\mu = (k^2-2)(\lambda\bar{u}_1 - i\beta\bar{u}_2) + k^2\bar{u}_{3,3} \tag{22}$$

where $k^2 = \kappa_b^2/\kappa_a^2$. Finally, the transformed boundary conditions are

$$\bar{\sigma}_{i3}(\lambda, 0^{\pm}) = \mu R_i(\lambda), \tag{23}$$

where R_i is an unknown function analytic in the right half of the λ-plane. Taking R_i to be independent of whether $z = 0$ is approached through positive or negative values implies that the traction is continuous across $z = 0$ for $x > 0$.

The system of Wiener–Hopf equations is obtained by substituting (17) and (23) into (20)–(22), and the result is

$$R_i(\lambda) = \Psi_{ij}(\lambda)\overline{\Delta u}_j(\lambda)/2q\kappa_b^2 \tag{24}$$

where

$$\Psi_{11} = (4\lambda^2 pq - 4\lambda^2 q^2 - \kappa_b^2\lambda_b^2) \tag{25a}$$

$$\Psi_{12} = (-4pq + 4q^2 + \kappa_b^2)i\beta\lambda \tag{25b}$$

$$\Psi_{13} = \Psi_{31} = \Psi_{23} = \Psi_{32} = 0 \tag{25c}$$

$$\Psi_{21} = \Psi_{12} \tag{25d}$$

$$\Psi_{22} = (-4\beta^2 pq + 4\beta^2 q^2 + \kappa_b^4 + \kappa_b^2\lambda^2) \tag{25e}$$

$$\Psi_{33} = q\mathrm{d}(\lambda)/p \tag{25f}$$

$$p = (\lambda_a^2 - \lambda^2)^{\frac{1}{2}}, \qquad q = (\lambda_b^2 - \lambda^2)^{\frac{1}{2}} \tag{26}$$

$$\mathrm{d}(\lambda) = 4(\lambda^2 - \beta^2)pq + (2\lambda^2 + \kappa_b^2 - 2\beta^2)^2. \tag{27}$$

The branches of p and q selected are those for which $\mathrm{Re}\,(p) \geq 0$ and $\mathrm{Re}\,(q) \geq 0$ everywhere in the λ-plane. The function $\mathrm{d}(\lambda)$ is the modified Rayleigh wave function introduced in [1]. The equation $\mathrm{d}(\lambda) = 0$ has two roots in the cut λ-plane at $\lambda = \pm i\alpha$. These roots are indicated in Fig. 2.

It is helpful at this point to write $\overline{\Delta u}_i$ as a sum of two terms

$$\overline{\Delta u}_i(\lambda) = L_i(\lambda) - A_i/(\lambda + i\alpha), \tag{28}$$

which separates out the discontinuity in displacement due to the incident wave. The function L_i is the transform of the discontinuity in u_i across $z = 0$ due to outgoing waves only and this function is, therefore, analytic in the left half-plane. It is also convenient to introduce the auxiliary function $D(\lambda)$ defined by

$$D(\lambda) = \mathrm{d}(\lambda)/\kappa(\lambda^2 + \alpha^2), \tag{29}$$

where $\kappa = 2(\kappa_b^2 - \kappa_a^2)$. This function has neither zeros nor poles in the cut λ-plane, and $D \to 1$ as $|\lambda| \to \infty$. A product factorization of D into sectionally analytic functions D_+ and D_-, in a form which is useful here, has been presented in [1]. As usual, the subscript plus or minus denotes the domain of analyticity, the plus (minus) referring to the right (left) half-plane.

The three uncoupled Wiener–Hopf type equations are now obtained from (24). In view of (25c), one of these equations is obtained by setting $i = 3$ in (24), with the result

$$2\kappa_b^2 pR_3(\lambda) = \overline{\Delta u}_3(\lambda)\,\mathrm{d}(\lambda). \tag{30}$$

The second of these equations is obtained by taking the sum of $i\beta$ times (24) with $i = 1$ and λ times (24) with $i = 2$, with the result

$$2[i\beta R_1(\lambda) + \lambda R_2(\lambda)] = -q[i\beta\overline{\Delta u}_i(\lambda) + \lambda\overline{\Delta u}_2(\lambda)]. \tag{31}$$

Clearly, the left side of (31) is a function which is analytic in the right half-plane. Furthermore, the quantity in parenthesis on the right side of (31) is the sum of a function which is analytic in the left half-plane plus a function whose only singularity is a simple pole at

$\lambda = -i\alpha$. The relation (31) is thus a very common form of a Wiener–Hopf equation. Finally, the third equation is obtained by taking the sum of λ times (24) with $i = 1$ plus $-i\beta$ times (24) with $i = 2$, with the result

$$2\kappa_b^2 q[\lambda R_1(\lambda) - i\beta R_2(\lambda)] = -\mathrm{d}(\lambda)[-\lambda\overline{\Delta u}_1(\lambda) + i\beta\overline{\Delta u}_2(\lambda)]. \tag{32}$$

Relation (32) has essentially the same form as (31).

The steps which are followed in determining $\overline{\Delta u}_3$ from (30) are identical to those discussed in [4], and only the result is included here,

$$\overline{\Delta u}_3 = \frac{A_3(2i\alpha)p_-(\lambda)D_-(-i\alpha)}{(\lambda^2+\alpha^2)p_-(-i\alpha)D_-(\lambda)}. \tag{33}$$

To derive (33) it is necessary to factor p into a product of sectionally analytic function p_+p_-, where the plus and minus signs indicate the same domains of analyticity as before. The result of the factorization is

$$p_\pm(\lambda) = (\lambda_a \pm \lambda)^{\frac{1}{2}}, \tag{34}$$

with a similar result holding for q.

In the initial stages of solution, the remaining two Wiener–Hopf equations may be approached separately and in the usual way, for example, as outlined in [4]. Certain functions are factored into sectionally analytic functions and each equation is rewritten in such a way that opposite sides of the equation are analytic functions in complementary half-planes. By an analytic continuation argument, it is concluded that each side is one and the same entire function. It can often be shown by applying "uniqueness conditions", as is the case here, that the entire function has algebraic behavior of a certain order at infinity and is therefore a polynomial function of known order. The problem then remains of determining the unknown coefficients of this polynomial function.

For the problem being considered here, it turns out that the polynomials obtained for both (31) and (32) are simply constants. This follows from imposition of the uniqueness condition requiring the displacement to be continuous and the strain energy to be integrable at the edge of the crack. It is found that, in order to determine the values of these constants, the two equations must be considered together. Since this feature of interaction is the only novel aspect of the solution of the problem at hand, only those steps relating to this feature are outlined.

If the steps outlined above are carried out for equations (31) and (32), the following results are obtained,

$$i\beta\overline{\Delta u}_1 + \lambda\overline{\Delta u}_2 = -E_1/q_-(\lambda), \tag{35}$$

$$\lambda\overline{\Delta u}_1 - i\beta\overline{\Delta u}_2 = \frac{q_-(\lambda)}{(\lambda^2+\alpha^2)D_-(\lambda)}\{(\lambda+i\alpha)E_2 - K\}, \tag{36}$$

$$K = -2\alpha\Gamma A_3 D_-(-i\alpha)/q_-(-i\alpha), \tag{37}$$

where E_1 and E_2 are the aforementioned unknown constants. Together (35) and (36) form a system of linear algebraic equations for $\overline{\Delta u}_1$ and $\overline{\Delta u}_2$. The determinant of the coefficients of this system is $(\beta^2 - \lambda^2)$. Therefore, for all values of λ such that $\lambda^2 \neq \beta^2$ the system of equations has a unique nontrivial solution. When $\lambda^2 = \beta^2$, however, the system of equations has no solution, in general. This undesirable result can be avoided by requiring that the coefficient matrix and the augmented matrix of the system have the same rank for all λ and, in particular, for $\lambda = \pm\beta$. This is essentially a compatibility condition on the right sides of (35) and (36) which requires that, for $\lambda = \pm\beta$, the right side of (35) must equal $\pm i$ times the right side of (36). This yields two equations which are linear in E_1 and E_2 with known constant coefficients. The solution of these equations completes the solution of the Wiener–Hopf problem.

The constants E_1 and E_2 may be found by an alternate scheme. The equations (35) and (36) are first solved for $\overline{\Delta u_1}$ and $\overline{\Delta u_2}$. The solution shows that both $\overline{\Delta u_1}$ and $\overline{\Delta u_2}$ have apparent simple poles at $\lambda = \pm\beta$. From (28), it is clear that neither of these functions may have a pole at $\lambda = -\beta$, since the only admissible singularity in the left half-plane is at $\lambda = -i\alpha$. Furthermore, the presence of a pole at $\lambda = \beta$ implies that the discontinuity in displacement across $z = 0$ increases exponentially with distance from the edge of the crack. Such a result is physically inadmissible. The net result is that the residues of the apparent poles of $\overline{\Delta u_1}$ and $\overline{\Delta u_2}$ at $\lambda = \pm\beta$ must vanish. This yields four equations, only two of which are independent, for the determination of E_1 and E_2. These equations are, of course, the same as the equations referred to in the preceding paragraph. The result of solving them is

$$E_1 = 2i\beta\kappa_b^2 K/\gamma^2 C, \tag{38}$$

$$E_2 = K[D_+(\beta)(\lambda_b-\beta)+D_-(\beta)(\lambda_b+\beta)]/C, \tag{39}$$

$$C = [D_+(\beta)(\lambda_b-\beta)(\beta+i\alpha)-D_-(\beta)(\lambda_b+\beta)(\beta-i\alpha)]. \tag{40}$$

This essentially completes the outline of the steps followed in determining the transformed displacements. The constants E_1 and E_2 given in (38) and (39) may be substituted into (35) and (36). The latter equations, along with (33), determine $\overline{\Delta u_i}$. The expressions for $\overline{\Delta u_i}$ may then be substituted into (17) which yields the Laplace transform of the displacement, which must be inverted. Since attention here is focused primarily on the reflected surface waves on $z = 0^\pm$, and since a surface wave is completely characterized by a single component of surface displacement only the z-component of surface displacement is considered here. The result of the analysis outlined above is

$$\bar{u}_3(\lambda, 0^\pm) = \frac{i\alpha A_3 D_-(-i\alpha)}{(\lambda^2+\alpha^2)D_-(\lambda)}\left\{\pm\frac{p_-(\lambda)}{p_-(-i\alpha)}+\frac{i\Gamma(2pq-2q^2-\kappa_b^2)}{\kappa_b^2 q_+(\lambda)q_-(-i\alpha)}[(\lambda+i\alpha)E_2/K-1]\right\}. \tag{41}$$

The displacement in the z-direction on $z = 0^\pm$ is then found by inverting (41),

$$u_3(x, 0^\pm) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \bar{u}_3(\lambda, 0^\pm)\,\mathrm{e}^{\lambda x}\,\mathrm{d}\lambda \tag{42}$$

where the inversion path is shown in Fig. 2. The integrand of (41) has branch points at $\lambda = \pm\lambda_a, \pm\lambda_b$ and simple poles at $\lambda = \pm i\alpha$. For $x > 0$, the path of integration of (42) may be deformed into the left half-plane, with the result that $u_3(x, 0^\pm)$ may be expressed as the sum of the residue at $\lambda = -i\alpha$ plus a branch line integral. The latter represents the body waves in $x > 0$ arising from diffraction. For $z = 0^+$, the residue represents a surface wave which exactly cancels the incident surface wave for $x > 0$, while for $z = 0^-$, the residue is identically zero. For $x < 0$, the path of integration may be deformed into the right half-plane, with the result that u_3 may be expressed as the sum of the residue at $\lambda = i\alpha$ plus a branch line integral. As before, the latter represents the body waves arising from diffraction. The residue, on the other hand, represents the reflected surface waves on $z = 0^{\pm}$ which are the waves of interest here. Denoting the residue by $u_3^s(x, 0^{\pm})$, the reflected surface wave amplitudes are given explicitly by

$$u_3^s(x, 0^{\pm}) = \frac{1}{2} A_3 \frac{D_-(-i\alpha)}{D_-(i\alpha)} \left\{ \pm \frac{p_-(i\alpha)}{p_-(-i\alpha)} - \frac{q_-(i\alpha)}{q_-(-i\alpha)} (2i\alpha E_2/K - 1) \right\} \mathrm{e}^{i(\pi + \alpha x)}. \tag{43}$$

All quantities in (43) have been explicitly defined except the ratio $D_-(-i\alpha)/D_-(i\alpha)$, which is given in [1]. The reflected surface waves are then described by the real parts of

$$U_3(x, y, 0^{\pm}, t) = u_3(x, 0^{\pm}) \mathrm{e}^{i(\omega t - \beta y)}. \tag{44}$$

NUMERICAL RESULTS AND DISCUSSION

As has been shown, the transformed solution of this problem is quite complicated and only expressions for the surface wave contribution can be extracted for direct numerical evaluation. The qualitative features of the complete wave motion associated with the reflection process are apparent from the analysis and the numerical results, however. For any angle of incidence θ the incident surface wave gives rise to reflected surface waves on $z = 0^{\pm}$. The situation is more complex than the elementary reflection process, however, because of the possibility of mode conversion from the surface wave mode to a body wave mode upon reflection. If such mode conversion occurs, then the energy carried away from the edge of the crack in the form of surface waves is less than the energy transported to the edge by the incident surface wave. In contrast, the elementary reflection process is governed by a conservation of energy at the reflecting boundary.

The ratios of the amplitudes of the reflected surface waves to the amplitude of the incident wave versus angle of incidence has been calculated for the special case of Poisson's ratio of 0·25. The result is shown in Fig. 3. The wavenumbers for this case are related by $\kappa_b^2 = 3\kappa_a^2$ and $\gamma^2 = 3{\cdot}549\kappa_a^2$. Figure 4 shows the phases of the reflected surface waves, assuming the phase of the incident wave to be zero. A parameter which is a convenient measure of the relative energy of a surface wave can be obtained by squaring the magnitude of the z-component of displacement of the surface wave. Assuming the energy of the incident wave to be unity, the relative energy of each of the reflected surface waves is shown in Fig. 5. The sum of these relative energies is also shown in Fig. 5.

For small values of θ the apparent wavenumber β of the incident surface wave along the edge of the crack is greater than the larger wavenumber of body waves. Equivalently, the apparent wave speed of the incident wave along the edge ω/β is less than the slower body wave speed. Consequently, only localized, nonpropagating body wave modes are excited near the crack tip. All energy transported to the edge by the incident wave must be carried away by the reflected surface waves. This is indeed the case, as can be seen from Fig. 5. Even though the partition of energy of the two reflected surface waves varies with θ, the sum of the energies is unity for sufficiently small θ, that is, for $\beta > \kappa_b$. For the numerical values used here $\beta = \kappa_b$ at $\theta = 23{\cdot}2°$, which is indicated in the figures.

When $\beta < \kappa_b$, the apparent signal speed along the edge of the crack is greater than the shear wave speed. Propagating shear wave modes are therefore excited, which carry some energy away from the edge of the crack. This is born out by Fig. 5, which shows that for $\beta < \kappa_b$ the total energy of the reflected waves is less than the energy of the incident wave.

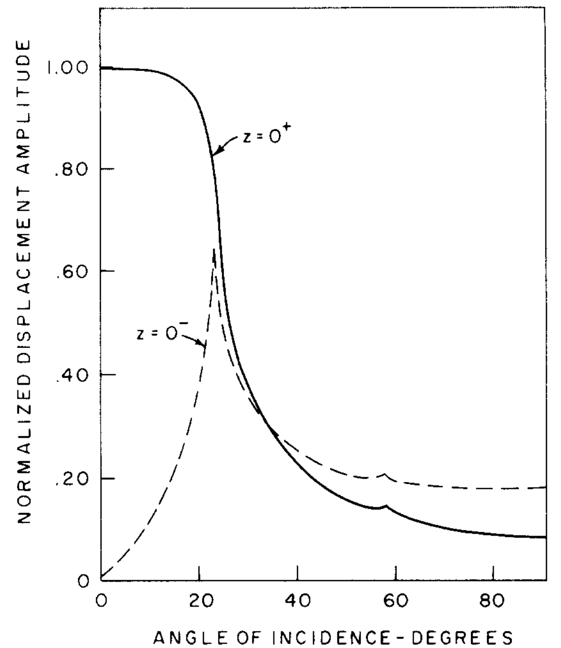

FIG. 3. Normalized amplitudes of reflected surface waves vs. angle of incidence.

The surfaces of constant phase of these shear waves are right circular cones whose axes coincide with the y-axis. When $\beta < \kappa_a$, propagating dilatational modes are also excited. The value of θ at which $\beta = \kappa_a$ is approximately 57·9°.

In this paper, only an incident surface wave on $z = 0^+$ has been considered. Results for other cases may be easily deduced from results for this case, however. For example,

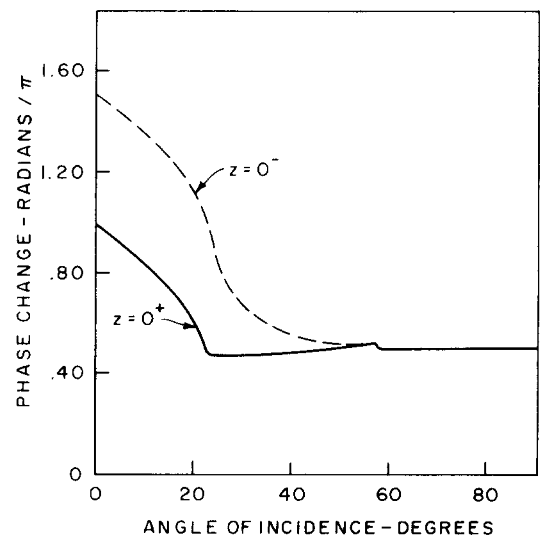

FIG. 4. Phase change of reflected surface waves vs. angle of incidence.

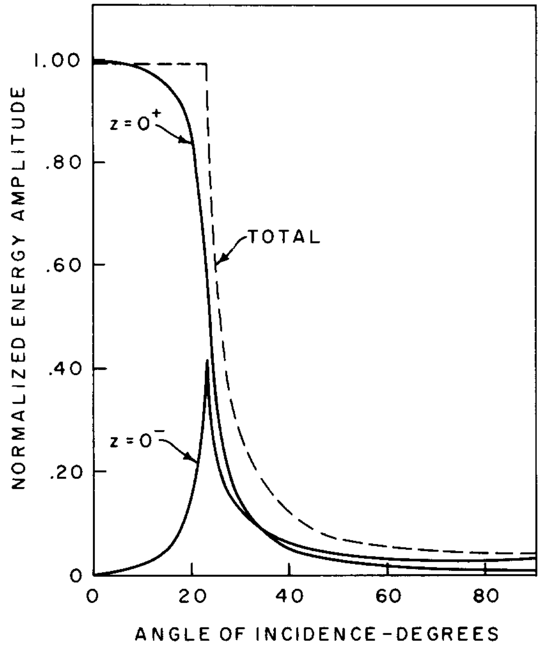

FIG. 5. Normalized energy of reflected surface waves vs. angle of incidence.

suppose a surface wave is propagating on $z = 0^-$ and is obliquely incident on the edge of the crack with angle of incidence θ. Furthermore, suppose that the z-component of displacement of the incident wave is the same as for the original problem. This does not impose a restriction on the modified problem, but merely fixes the zero phase line so that a direct correlation can be made. The results for the modified problem may then be obtained by replacing A_2 and A_3 by $-A_2$ and $-A_3$. Then, in view of (5), the reflected surface waves in the modified problem are given by (43) with the sign of the first term in brackets being changed. That is, if $v_i(x, z)$ is the solution of the modified problem, then

$$v_i^s(x, 0^\pm) = u_i^s(x, 0^\mp). \tag{45}$$

Similarly, if the z-component of the displacement of the incident wave on $z = 0^-$ is π radians out of phase, then the solution is obtained by replacing A_3 by $-A_3$ in the original problem. If $w_i(x, z)$ is the solution of this second modified problem, then

$$w_i^s(x, 0^\pm) = -u_i^s(x, 0^\mp). \tag{46}$$

Any symmetric or antisymmetric motions (with respect to $z = 0$) may be constructed from (45) and (46). For example, for $\theta = 90°$ the sum $u_i^s(x, 0) + w_i^s(x, 0)$ yields the reflection coefficient by Fredricks and Knopoff [8] for a Rayleigh wave normally incident on a rigid smooth barrier loading half of the surface of a half-space.

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Абстракт—Исследется отображение свободной поверхностной волны Релея от края полуплоской щели, в неограниченном изотропном упругом меле. Тармоническая по времени, поверхностная волна, распространяющаяся по поверхности щели, косо направлена на краю так, что задача оказывается трехмерной. Применение задачи в перемещениях сводит решение проблемы к решению интегрального уравнения. Это уравнение решается с помощью методов преобразования Лапласа и метода Винера-Хопфа. Преобразованное перемещение имеет проетой полюс, соотвествующий отображенным поверхностным волнам. Определяются вычеты. Подсчитано численно амплитуды и фазы отображенных, поверхностных волн на двух сторонах щели и представляются графически, в зависимости от угла ла падения. Указываетя, также, на графиках энергии отображенных волн.